TWISTED PARTIAL ACTIONS ON SEMIPRIME RINGS

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The multiplier ring

Let A be a non-necessarily unital k-algebra. The multiplier ring $\mathcal{M}(A)$ of A is defined as the set

$$\mathcal{M}(A) = \{(R, L) \in End({}_A A) \times End(A_A) : (aR)b = a(Lb) \forall a, b \in A\}$$

with the following operations:

(i) $(R, L) + (R', L') = (R + R', L + L')$;

(ii) $(R, L) \circ (R', L') = (R' \circ R, L \circ L')$.

We use the usual notations for $R : {}_A A \longrightarrow {}_A A$ and $L : A_A \longrightarrow A_A$, respectively. That is, for $w = (R, L) \in \mathcal{M}(A)$ and $a \in A$ we set $aw = aR$ and $wa = La$.

Partial Actions

Definition Let G be a group and A a unital k-algebra, k a commutative ring. A **partial action** α **of** G **on** A is a collection of ideals S_g, $g \in G$, of R and isomorphisms of k-algebras $\alpha_g : S_{g^{-1}} \longrightarrow S_g$ such that:

(i) $S_1 = R$ and α_1 is the identity mapping on R.

(ii) $S_{(gh)^{-1}} \supseteq \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.

(iii) $\alpha_g \circ \alpha_h(x) = \alpha_{(gh)}(x)$, for any $x \in \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.

More generally we have

Twisted Partial Actions

Definition A **twisted partial action** α **of** G **on** A is a collection of ideals S_g, $g \in G$, of A,

isomorphisms $\alpha_g : S_{g^{-1}} \longrightarrow S_g$, and a family $\{w_{g,h}\}_{(g,h)\in G\times G}$,

where, for each $(g,h) \in G \times G$, $w_{g,h}$ is an invertible element from $\mathcal{M}(S_g S_{gh})$ satisfying the following properties, for all $g, h, t \in G$:

(i) $S_g^2 = S_g$, $S_g S_h = S_h S_g$;

(ii) $S_1 = A$ and α_1 is the identity mapping of A;

(iii) $\alpha_g(S_{g^{-1}} S_h) = S_g S_{gh}$;

(iv) $\alpha_g \circ \alpha_h(a) = w_{g,h} \alpha_{gh}(a) w_{gh,t}^{-1}$, for every $a \in S_{h^{-1}} S_{h^{-1}g^{-1}}$;

(v) $w_{g,1} = w_{1,g} = id$;

(vi) $\alpha_g(a w_{h,t}) w_{g,ht} = \alpha_g(a) w_{g,h} w_{gh,t}$, for every $a \in S_{g^{-1}} S_h S_{ht}$.

Note that α is a twisted global action if $S_g = A$, for all $g \in G$.

Example Given a twisted global action

$$\beta = (B, \{\beta_g\}_{g \in G}, \{u_{g,h}\}_{(g,h) \in G \times G})$$

of G on a non-necessarily unital ring B,

One can restrict β to a two-sided ideal A of B which has an identity 1_A as follows:

Put $S_g = A \cap \beta(A) = A.\beta(A)$. Then S_g is a two-sided ideal of A which has an identity $1_g = 1_A \beta_g(1_A)$.

Also define $\alpha_g = \beta|_{S_{g^{-1}}}$ and $w_{g,h} = u_{g,h} 1_A \beta_g(1_A \beta_{g^{-1}h}(1_A))$.

Then it is easy to see that this gives a twisted partial action of G on A.

Definition of Enveloping action

A twisted global action $\beta = (B, \{\beta_g\}_{g\in G}, \{u_{g,h}\}_{(g,h)\in G\times G})$ of G on B is said to be a *globalization* (or an *enveloping action*) for the partial action α of G on A if there exists a monomorphism $\psi : A \longrightarrow B$ such that:

(i) $\psi(A)$ is an ideal of B;

(ii) $B = \sum_{g\in G} \beta_g(\psi(A))$;

(iii) $\psi(S_g) = \psi(A) \cap \beta_g(\psi(A))$, for any $g \in G$;

(iv) $\psi \circ \alpha_g = \beta_g \circ \psi$ on $S_{g^{-1}}$, for any $g \in G$;

(v) $\psi(aw_{g,h}) = \psi(a)u_{g,h}$, $\psi(w_{g,h}a) = u_{g,h}\psi(a)$, for any $g, h \in G$, $a \in S_g S_{gh}$.

We say that β is a **weak enveloping action** of α if the items (i), (iv) and (v) of the above definition are satisfied.

The following result was proved by M. Dokuchaev, R. Exel and J. J. Simon.

Theorem Let A be a unital ring which is a (not necessarily finite) product of indecomposable rings. A twisted partial action α of G on A has a globalization if and only if each ideal S_g, $g \in G$, is a unital ring.

Note that when α is a partial action, then the assumption that A is a product of indecomposable rings is not necessary.

Partial crossed product

Let α be a twisted partial action of G on R. The **twisted partial crossed product**

$A \star_\alpha G$ is defined as the direct sum $\oplus_{g \in G} S_g \delta_g$,

in which the $\delta_g' s$ are symbols and the multiplication is defined by the rule:

$$(a_g \delta_g).(b_h \delta_h) = \alpha_g(\alpha_G^{-1}(a_g) b_h) w_{g,h} \delta_{g,h}.$$

Here $w_{g,h}$ acts as right multiplier on $\alpha_g(\alpha_g^{-1}(a_g) b_h) \in S_g S_{gh}$.

The associativity of the twisted partial crossed product was proved by the same authors mentioned above.

Semiprime Rings

We know that a semiprime ring is a ring which does not have nilpotent ideals.

Assume that A is a semiprime ring. Given an ideal I of A, the closure $[I]$ of I in A is defined as:

$$[I] = \{x \in A : xH \subseteq I, \text{ for an essential ideal } H \text{ of } A\}$$

We have that $[I]$ is also an ideal and $I \subseteq [I]$. The ideal I is said to be closed if $[I] = I$.

Rings of quotients

It is well-known that for any semiprime ring A there exists a Martindale left ring of quotients Q and a maximal left ring of quotients Q_m of A. Both rings of quotients are also semiprime and we have $A \subseteq Q \subseteq Q_m$.

Assume that A is semiprime and Q is the Martindale (left) ring of quotients of A. If I is an ideal of A, the closure of I in Q is defined as:

$$I^* = \{q \in Q : qH \subseteq I, \text{ for an essential ideal } H \text{ of } A\}$$

Then I^* is a closed ideal of the semiprime ring Q and there exists a central idempotent $e \in Q$ such that $I^* = Qe$.

Theorem. There exists a one-to-one correspondence between the closed ideals of A and the closed ideals of Q. This correspondence associates the closed ideal I of A with the closed ideal L of Q if $L \cap A = I$. In this case $L = I^*$ is generated by a central idempotent of Q.

The same result holds for the relations between closed ideals of A and closed ideals of Q_m.

Extensions of Twisted Partial Actions on Semiprime Rings

Hereafter we assume that A is a semiprime ring, Q is the Martindale left ring of quotients of A and Q_m is the maximal left ring of quotients of A. Also

$$\alpha = (\{S_g\}_{g\in G}, \{\alpha_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G})$$

is a twisted partial action of G on A.

We proved that the twisted partial action α can be extended to Q. We have the following:

Theorem. For the given α there exists a twisted partial action

$$\alpha^* = (\{S_g^*\}_{g\in G}, \{\alpha_g^*\}_{g\in G}, \{w_{g,h}^*\}_{(g,h)\in G\times G})$$

of G on Q such that $\alpha_g^*|S_{g^{-1}} = \alpha_g$ and $w_{g,h}^*|_{S_g S_{gh}} = w_{g,h}$, for all $g, h \in G$.

It was more difficult to find a way to prove similar result for extending partial actions to Q_m. But using the fact that all the ideals S_g^* are generated by central idempotents finally we find a way to prove that this is also true. We have:

Theorem Let $\alpha = (\{S_g\}_{g\in G}, \{\alpha_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G})$ be a twisted partial action of G on a semiprime ring A. Then there exists a twisted partial action

$$\alpha^{**} = (\{S_g^{**}\}_{g\in G}, \{\alpha_g^{**}\}_{g\in G}, \{w_{g,h}^{**}\}_{(g,h)\in G\times G})$$

of G on Q_m such that $\alpha_g^{**}|S_{g^{-1}} = \alpha_g$ and $w_{g,h}^{**}|_{S_g S_{gh}} = w_{g,h}$, for all $g, h \in G$.

This additional extension to Q_m was essential for our purposes. In particular for applying the results to study partial crossed products on semiprime Goldie rings

Let α be a partial action (without twisting) of G on a semiprime rings A we can consider the extension α^* of α to Q. Since all the ideals S_g^* are generated by central idempotents then there exists an enveloping action (B, β) of α^*. Thus we have:

$$(A, \alpha) \hookrightarrow (Q, \alpha^*) \hookrightarrow (B, \beta).$$

Hence there exists a global action (B, β) extending α. This is what we called a **weak enveloping action** of α.

Corollary Any partial action on a semiprime ring has a weak enveloping action.

Under some finiteness conditions the same result holds for twisted partial actions. In particular, this holds when A is a Goldie ring.

Main References

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